

INTRODUCTION TO CYCLIC HOMOLOGY

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1 Hochschild homology

The Hochschild homology arose naturally in the algebraic theory as a machine to compute the default of commutativity of an algebra. This homology defines a non-commutative generalization of the notion of differential forms. This analogy is known as the Hochschild-Kostant-Rosenberg theorem and propelled the domain of non-commutative Geometry in the early 60's. A complete introduction is given in [Lod92, p. 1.1].

Let A be a complex unital algebra. We use $\otimes$ to denote the tensor product over the complex numbers. For $n \geq 0$, consider the vector space $C_n(A) := A^{\otimes n+1}$. One can define $n+1$ operators denoted $d_i : C_n(A) \rightarrow C_{n-1}(A)$, for $0 \leq i \leq n$, given by:

$$d_i(a_0 \otimes \cdots \otimes a_n) := \begin{cases} a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n & i = 0 \\ a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & 0 < i < n \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & i = n \end{cases}$$

These operators verify the identities $d_i d_j = d_{j-1} d_i$ for $0 \leq i < j \leq n$, then induce a differential map

$$b := \sum_{i=0}^n (-1)^i d_i : C_n(A) \rightarrow C_{n-1}(A). \quad (1)$$

verifying the condition $b \circ b = 0$.

Definition 1.1. We define the **Hochschild homology of a complex unital algebra** A as the complex vector spaces

$$HH_*(A) := H_*(C_*(A), b).$$

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The Hochschild homology groups are functorial for the variable in the sense that any homomorphism of complex unital algebras $f : A \rightarrow B$ induces a linear map $HH(f) : HH_*(A) \rightarrow HH_*(B)$ at any degree.

Exercise 1 Show that for any unital algebra A , there exists an isomorphism $HH_0(A) \simeq A/[A, A]$ where $[A, A] = \{ab - ba \mid a, b \in A\}$ denotes the space of commutators of the algebra. Conclude that $HH_0(A) \simeq A$ if and only if A is commutative.

The Hoshchild homology can be defined using coefficients. The coefficient space requires a right and left actions of A to ensure the differential b is well-defined. Such a space is called a bimodule over A . We define the Hochschild homology of a complex algebra A with **coefficients in a A -bimodule** M as the homology of the chain complex made of the chains $C_n(A, M) := M \otimes A^{\otimes n}$ and the differential b as before. Its homology groups are denoted $HH_*(A, M)$ and functorial in both variables. We recover $HH_*(A)$ as the Hochschild homology $HH_*(A, A)$ with coefficients in the A -bimodule A itself. One can find this construction in many book about homological algebra.

Remark There exists a standard way to extend a functor R from unital algebras with values in vector space to the category of not necessarily unital algebras. The idea is the following. Given a complex *non-unital* algebra A one can adjoined to it a unit by considering the algebra $A_+ := \mathbb{C} \oplus A$ endowed with the product $(\lambda, a) \cdot (\lambda', a') = (\lambda\lambda', \lambda a' + a\lambda' + aa')$. By definition, the extension of R to non necessarily unital algebra is given by

$$R(A) := \text{Coker}(R(\mathbb{C}) \longrightarrow R(A_+))$$

which is well defined by functoriality. We will then mainly define the homologies we will encounter for unital algebras and use this process to extends their definition to the category of non necessarily unital algebra.

Remark The Hochschild homology corresponds to the *derived functor* [GM02] of the left-exact functor $A \mapsto A/[A, A]$, or in other words to the Tor-functor $HH_*(A) \simeq \text{Tor}_*^{A\text{-bimod}}(A, A)$. This point of view allows us to compute this homology using other chain-complexes. Whenever $P_*(A)$ is a projective resolution of A as a A -bimodule, its Hochschild homology is equal to $HH_*(A) \simeq H_*(A \otimes_{A\text{-bimod}} P_*(A))$ and the result doesn't depend on the resolution.

In the case of topological algebras we must specify which particular topological

tensor product we use to compute Hochschild theory. From now to the end, we will mostly work with **complex complete locally convex algebras**. This class of algebras includes Banach algebras, C^* -algebras, Fréchet algebras and many smooth function algebras as *e.g.* of Schwartz-type. In this setup, the completed projective tensor product preserves the underlying topologies unlike the complex tensor product, used until now. Given two Banach vector spaces E and F , their **completed projective tensor product** is denoted $E \widehat{\otimes}_\pi F$ and given by the completion of $E \otimes F$ with respect to the largest semi-norm $\|\cdot\|_\pi$ such that $\|e \otimes f\|_\pi = \|e\|_E \cdot \|f\|_F$ on elementary tensors [Gro54]. With two compact manifolds M and N , the completed projective tensor product realizes the identity

$$\mathcal{C}^\infty(M) \widehat{\otimes}_\pi \mathcal{C}^\infty(N) \simeq \mathcal{C}^\infty(M \times N).$$

Definition 1.2. We define the **Hochschild homology of a complex locally convex algebra** A as $HH_\star(A) := H_\star(C_\star(A, \widehat{\otimes}_\pi), b)$, the homology of its Hochschild complex defined using complete projective tensor product.

Let us compute some examples:

- $A = \mathbb{C}$: The zero-th homology groups has already been computed in exercise 1 and all the other homology groups are zero:

$$HH_n(\mathbb{C}) \simeq \begin{cases} \mathbb{C} & \text{if } n = 0; \\ 0 & \text{if } n > 0. \end{cases}$$

- $A = \mathbb{C}[T]$: Again by exercise 1, we know that $HH_0(\mathbb{C}[T]) \simeq \mathbb{C}[T]$. We can show that $HH_1(\mathbb{C}[T]) \simeq \mathbb{C}[T]$ and $HH_n(\mathbb{C}[T]) = 0$ for any $n > 1$. More generally, we have

$$HH_n(\mathbb{C}[T_1, \dots, T_r]) = 0 \text{ for } n > r.$$

- $A = \mathbb{C}[\Gamma]$ is the convolution algebra of a discrete group Γ , i.e. its space of Dirac measures with pointwise product. Its Hochschild homology recover the homology of the group [Mac63].

Theorem 1.3 (Mac Lane). *There exists an isomorphism*

$$HH_\star(\mathbb{C}[\Gamma]) \simeq H_\star(\Gamma, \mathbb{C}) \simeq \bigoplus_{\langle \gamma \rangle \in (\Gamma)} H_\star(B\Gamma_\gamma, \mathbb{C})$$

where the direct sum runs along the conjugacy classes of the group and $B\Gamma_\gamma$ is the classifying space of the the centralizer Γ_γ of γ in Γ .

- $A = \mathcal{C}^\infty(M, \mathbb{C})$ is the algebra of smooth functions over a compact manifold: The description of its Hochschild complex in terms of differential forms on M yields to the following important theorem [HKR09].

Theorem 1.4. ★ (Hochschild-Kostant-Rosenberg) *Let M be a compact manifold. The Hochschild homology of $\mathcal{C}^\infty(M) := \mathcal{C}^\infty(M, \mathbb{C})$ is isomorphic to the graded-algebra of differential forms over M*

$$HH_\star(\mathcal{C}^\infty(M)) \simeq \Omega^\star(M).$$

Furthermore, the isomorphism is given by the anti-symmetrization map

$$\epsilon : f_0 \otimes \dots \otimes f_n \mapsto f_0 df_1 \dots df_n. \quad (2)$$

2 Cyclic (co)homology

Although Hochschild homology plays the role of differential forms over a manifold, it lacks the important information of the De Rham differential. In 1963, G. Rinehart proposed a generalization of the De Rham cohomology which fits in the Hochschild-Kostant-Rosenberg theorem above [Rin63]. A. Connes in his book [Con85] defined an operator

$$HH_n(A) \xrightarrow{B} HH_{n+1}(A)$$

whose aim is to play the role of the De Rham differential in a non-commutative framework. Through the anti-symmetrization map ϵ defined in (2) above, it becomes a multiple of the exterior derivative on the De Rham complex

$$\begin{array}{ccc} HH_n(\mathcal{C}^\infty(M)) & \xrightarrow{B} & HH_{n+1}(\mathcal{C}^\infty(M)) \\ \epsilon \downarrow \sim & & \sim \downarrow \epsilon \\ \Omega^n(M) & \xrightarrow{(n+1) \cdot d} & \Omega^{n+1}(M) \end{array} \quad (3)$$

The differential b together with the operator B (which is a differential) gives naturally rise to the study of a bicomplex whose homology is called *cyclic homology* and computes a non-commutative De Rham cohomology.

Fix A to be a unital (complete locally convex) algebra. The **Connes' boundary operator** $B : C_n(A) \longrightarrow C_{n+1}(A)$ is defined at the level of chain-complex

as follows

$$B(a_0 \otimes \cdots \otimes a_n) := \sum_{i=0}^n \left[(-1)^{ni} 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1} \right. \\ \left. - (-1)^{n(i-1)} a_{i-1} \otimes 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-2} \right].$$

Remark In [Lod92, p. 2.1.7] you can find an alternative definition of the Connes' boundary map using the cyclic operator $t : C_n(A) \rightarrow C_n(A)$ given by $t(a_0 \otimes \cdots \otimes a_n) = a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$. It is the historical definition and justifies the term "cyclic" in the name cyclic homology.

Proposition 2.1. *The Connes' boundary map verifies the properties*

$$B \circ B = 0 \quad \text{and} \quad Bb + bB = 0.$$

The chain-complex $C_\star(A)$ is endowed with the two commuting differentials b and B of degree -1 and +1 respectively, which yields to a chain bicomplex. Let us call **cyclic bicomplex** of a unital (complete locally convex) algebra the following

$$\mathcal{B}_\star(A) := \begin{array}{ccccc} & \downarrow b & & \downarrow b & & \downarrow b \\ C_2(A) & \xleftarrow{B} & C_1(A) & \xleftarrow{B} & C_0(A) & \xleftarrow{B} \\ \downarrow b & & \downarrow b & & & \\ C_1(A) & \xleftarrow{B} & C_0(A) & & & \\ \downarrow b & & & & & \\ C_0(A) & & & & & \end{array}.$$

Computation of the homology of a bicomplex One can associate to any bicomplex $(C_{\star,\star}, \partial, \partial')$ a chain-complex $\text{Tot } C_{\star,\star}$ called **total complex**. Its n -chains are given by $(\text{Tot } C_{\star,\star})_n := \bigoplus_{p+q=n} C_{p,q}$ and it is endowed with the differential $\partial + \partial'$. It can be viewed as a *diagonal* complex where you glue together the successive diagonals.

$$C_{\star,\star} = \begin{array}{ccccccc} & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & \\ C_{0,2} & \xleftarrow{\partial'} & C_{1,2} & \xleftarrow{\partial'} & C_{2,2} & \xleftarrow{\partial'} & \\ \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ C_{0,1} & \xleftarrow{\partial'} & C_{1,1} & \xleftarrow{\partial'} & C_{2,1} & \xleftarrow{\partial'} & \\ \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ C_{0,0} & \xleftarrow{\partial'} & C_{1,0} & \xleftarrow{\partial'} & C_{2,0} & \xleftarrow{\partial'} & \end{array}$$

$$\text{Tot } C_{\star,\star} = C_{0,0} \xleftarrow{\partial+\partial'} \bigoplus_{p+q=1} C_{p,q} \xleftarrow{\partial+\partial'} \bigoplus_{p+q=2} C_{p,q} \xleftarrow{\partial+\partial'} \cdots$$

Definition 2.2. The **cyclic complex** of a unital (complete locally convex) algebra A is defined to be the total complex of the cyclic bicomplex, denoted $CC_\star(A) := \text{Tot } \mathcal{B}_{\star,\star}(A)$. Its homology is called **cyclic homology** of A :

$$HC_\star(A) := H_\star(CC_\star(A)).$$

A way to compute cyclic homology is to use a long exact sequence involving both Hochschild and cyclic homology groups. The first column of the bicomplex $(\mathcal{B}_{\star,\star}(A), b, B)$ corresponds exactly to the chain-complex $(C_\star(A), b)$, computing the Hochschild homology. The image of the embedding $I : C_\star(A) \rightarrow \mathcal{B}_\star(A)$ then corresponds to the kernel of the two-degree shift $S : \mathcal{B}_\star(A) \rightarrow \mathcal{B}_\star(A)[-2]$ called **periodicity map**. The following sequence of complexes is then exact:

$$0 \longrightarrow C_\star(A) \xrightarrow{I} \mathcal{B}_\star(A) \xrightarrow{S} \mathcal{B}_\star(A)[-2] \longrightarrow 0. \quad (4)$$

By an argument of a long exact sequence we obtain the following statement we can find in [Con85].

Proposition 2.3 (Connes' exact sequence). *Given any unital (complete locally convex) algebra A , the following sequence is exact for all $n \geq 0$*

$$HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A)$$

Exercise 2 Let A be a (complete locally convex) algebra such that $HH_0(A) \simeq A$ and $HH_n(A) = 0$ for $n > 0$. Using the exact sequence above show that

$$HC_n(A) \simeq \begin{cases} A & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}.$$

Let us compute some examples:

- For any (complete locally convex) algebra A , the zeroth cyclic homology group is always

$$HC_0(A) \simeq HH_0(A) \simeq A/[A, A].$$

- $A = \mathbb{C}[\Gamma]$ is the convolution algebra of a discrete group: Given any $\gamma \in \Gamma$, the associated one-parameter group $\langle \gamma \rangle$ is a normal subgroup of the centralizer Γ_γ . Moreover, the homology of the classifying space of the group $\Gamma_\gamma / \langle \gamma \rangle$ reconstructs the cyclic homology of the convolution algebra [Lod92, p. 7.4.10].

Theorem 2.4. *There exists an isomorphism*

$$HC_\star(\mathbb{C}[\Gamma]) \simeq \bigoplus_{\langle \gamma \rangle \in \langle \Gamma \rangle} H_\star(B(\Gamma_\gamma / \langle \gamma \rangle), \mathbb{C}).$$

- $A = \mathcal{C}^\infty(M, \mathbb{C})$ is the algebra of smooth functions over a compact manifold: The bicomplex $CC_\star(\mathcal{C}^\infty(M))$ encapsulated both the data of differentials forms and of the exterior derivative. The observation of G. Rinehart in the early 60's is that the homology of its total complex needs to be related to the De Rham cohomology, which is the following theorem [Lod92, p. 2.3.7].

Theorem 2.5. *If M is a compact manifold, the cyclic homology of its algebra of smooth functions $\mathcal{C}^\infty(M)$ can be describe with De Rham cohomology groups*

$$HC_n(\mathcal{C}^\infty(M)) \simeq \Omega^n(M)/d(\Omega^{n-1}(M)) \oplus H_{dR}^{n-2}(M; \mathbb{C}) \oplus H_{dR}^{n-4}(M; \mathbb{C}) \oplus \dots$$

Exercise 3 In a straightforward computation, show that the diagram (3) commutes when M is a compact manifold (*i.e.* $\mathcal{C}^\infty(M)$ is unital). Remember that in the exterior algebra, the terms can be reorganized by multiplying with the sign of the permutation.

Theorem 2.6. *The Hochschild and cyclic homologies verify the following fundamental properties:*

- (Morita invariance) [Lod92, pp. 1.2.4, 2.2.9] *For any (complete locally convex) algebra A and $r \geq 0$, the trace map induces isomorphisms*

$$HH_\star(M_r(A)) \simeq HH_\star(A) \quad \text{and} \quad HC_\star(M_r(A)) \simeq HC_\star(A).$$

- (Excision) [Lod92, pp. 1.4.10, 2.2.17] *Given any unital (complete locally convex) algebra A and two-sided ideal $I \subseteq A$ such that $A \rightarrow A/I$ possesses a linear section, the following sequences are exact for all $n \geq 0$*

$$HH_n(I) \longrightarrow HH_n(A) \longrightarrow HH_n(A/I) \longrightarrow HH_{n-1}(I)$$

$$HC_n(I) \longrightarrow HC_n(A) \longrightarrow HC_n(A/I) \longrightarrow HC_{n-1}(I)$$

Why cyclic homology is not the best homology? Consider the topological space $\mathbb{R}^2$. It is smoothly contractible to the point so $\mathcal{C}^\infty(\mathbb{R}^2)$ and $\mathbb{C}$ are smoothly homotopy equivalent algebras. Surprisingly, their cyclic homology groups aren't isomorphic. Indeed, our previous computations show that

$$HC_1(\mathcal{C}^\infty(\mathbb{R}^2)) \simeq \Omega^1(\mathbb{R}^2)/d(\Omega^0(\mathbb{R}^2)) \quad \text{and} \quad HC_1(\mathbb{C}) = 0,$$

the first quotient space being non-trivial because the 1-form $\omega = xdy$ is not closed, then doesn't lie in the image of the exterior differential. Namely, one must remember that

Cyclic homology is not stable under smooth homotopy!

The result [Tab11] of G. Tabuada states that any homology theory which is Morita invariant, satisfies excision and stable under smooth homotopy must be $\mathbb{Z}/2\mathbb{Z}$ -graded. This is the reason why we introduce the periodic cyclic homology, a $\mathbb{Z}/2\mathbb{Z}$ -graded homology that respect all these properties and plays a relevant role for the study of K-theory.

3 Periodic cyclic homology

The K-theory is a fundamental tool in non-commutative geometry [Kar78]. Introduced in the 60's by Alexander Grothendieck in order to states the well known Grothendieck-Riemann-Roch theorem, it kept developing to appear as a major tool in the proofs of Bott periodicity and Atiyah-Singer theorems. A big issue of this theory is that the computation of these groups are highly non-trivial. The idea behind periodic cyclic homology is to propose a more computable homology verifying powerful properties and endowed with a pairing with K-theory.

We fix A to be a complete locally convex algebra as before. We can define the differentials $b : C_n(A) \rightarrow C_{n-1}(A)$ and $B : C_n(A) \rightarrow C_{n+1}(A)$ on the graded

vector space $C_\star(A)$ as above. These operators verify the property:

$$b^2 = B^2 = (b + B)^2 = 0.$$

We splits the Hochschild complex into the *even* and *odd* parts:

$$C_{\text{even}}(A) := \prod_{n \geq 0} C_{2n}(A) \quad \text{and} \quad C_{\text{odd}}(A) := \prod_{n \geq 0} C_{2n+1}(A).$$

The operator $b + B$ sends any even (*resp.* odd) degree element to a pair of odd (*resp.* even) degree elements, and then defines a differential map on the following $\mathbb{Z}/2\mathbb{Z}$ -graded complex.

Definition 3.1. We call **periodic cyclic complex of A** the following complex:

$$\widehat{CC}_\bullet(A) := C_{\text{even}}(A) \xrightleftharpoons[b+B]{b+B} C_{\text{odd}}(A)$$

The periodic cyclic complex has only two homology groups, the even and the odd, often called zeroth and first. We will distinguish the different type of graduations using the subscript " $\bullet$ " for any $\mathbb{Z}/2\mathbb{Z}$ -theories. Also, one might remark that the notation to designate the periodic cyclic complex corresponds to the notation for the cyclic complex with a hat. Indeed, it comes from a more general construction which associated to a $\mathbb{Z}$ -graded complex its periodic $\mathbb{Z}/2\mathbb{Z}$ -graded complex, here $\widehat{CC}_\bullet(A)$ is the periodization of $CC_\star(A)$.

Definition 3.2. We define the **periodic cyclic homology of A** as the homology of the periodic cyclic complex

$$HP_\bullet(A) := H_\bullet(\widehat{CC}_\bullet(A)).$$

To give intuition on this construction, one can sometimes view the periodic cyclic homology as the inverse limit of the cyclic homology groups. Indeed, when the periodicity map $S : HC_n(A) \longrightarrow HC_{n-2}(A)$ is surjective for all $n \geq 0$, then [Lod92, p. 5.1.10]

$$HP_\bullet(A) \simeq \varprojlim_i HC_{\bullet+2n}(A).$$

Exercise 4 Using exercise 2 show that if a (complete locally convex) algebra A verifies $HH_i(A) = 0$ for $i > 0$, then $HP_1(A) = 0$.

Theorem 3.3. ♣ The periodic cyclic homology respects the following properties:

- (Morita invariance) [LQ84] For any (complete locally convex) algebra A and $r \geq 0$, the trace map induces an isomorphism

$$HP_\bullet(M_r(A)) \simeq HP_\bullet(A).$$

- (Excision) [CQ97] [Pus01] Given any (complete locally convex) non-necessarily unital algebra A and two-sided ideal $I \subseteq A$ such that $A \rightarrow A/I$ possesses a linear section, the following is exact

$$\begin{array}{ccccc} HP_0(I) & \longrightarrow & HP_0(A) & \longrightarrow & HP_0(A/I) \\ \uparrow & & & & \downarrow \\ HP_1(A/I) & \longleftarrow & HP_1(A) & \longleftarrow & HP_1(I) \end{array}$$

- (Smooth homotopy invariance) [Kas87] If M is a compact manifold, the antisymmetrization map ϵ induces an isomorphism between periodic cyclic homology of its algebra of smooth functions $C^\infty(M)$ and its De Rham cohomology groups

$$HP_0(C^\infty(M)) \simeq \prod_{n \geq 0} H_{dR}^{2n}(M; \mathbb{C}) \quad \text{and} \quad HP_1(C^\infty(M)) \simeq \prod_{n \geq 0} H_{dR}^{2n+1}(M; \mathbb{C}).$$

In particular, the periodic cyclic homology is invariant under smooth homotopy.

In geometry, the **Chern classes** are topological invariants (sometimes complete) which furnish a way to distinguishing vector bundles over a smooth manifold. They all live in even degree De Rham cohomology. The **Chern character** is a character that gathers these data, associating to a vector bundle over a given smooth manifold a polynomial with coefficients in its Chern classes [Hir56]. It factorizes over the even K-group ring of the manifold as is transforms tensor product of vector bundles into cohomological product of their Chern classes

$$\text{Ch} : K^0(M) \longrightarrow \prod_{n \geq 0} H_{dR}^{2n}(M; \mathbb{C}).$$

The left hand space is an abelian group while the right hand one is a complex vector space. These are difficult to compare but while tensoring over the rational numbers, the following description occurs [Ati19].

Theorem 3.4. ♣ (Atiyah-Grothendieck-Hirzebruch) The Chern character is a rational isomorphism

$$\text{Ch}_{\mathbb{Q}} : K^0(M) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \prod_{n \geq 0} H_{dR}^{2n}(M; \mathbb{Q}).$$

As seen in the third point of theorem 3.3, the periodic cyclic homology of a unital algebra of smooth functions corresponds to the even and odd parts of the De Rham cohomology of its spectrum. This motivates the existence of an *algebraic version* of the Chern character from the K-theory of an algebra to its periodic cyclic homology. This would produce an algebraic computation for these unreachable K-groups.

Consider A to be a complete locally convex algebra. We recall that the set $\mathfrak{p}(A)$ of any-size idempotent matrices over A defines a monoid for the Whitney sum. The **even algebraic K-theory** of A is defined to be the Grothendieck group of this monoid

$$K_0(A) := \text{Gr}(\mathfrak{p}(A)).$$

It is generated by differences $[e] - [f] \in K_0(A)$ where e and f belongs to $\mathfrak{p}(A)$. Given an idempotent matrix $e \in M_r(A)$ and $n \geq 0$, let us write

$$\text{Ch}_n(e) := (-1)^n \frac{(2n)!}{n!} \text{Tr} \left(e - \frac{1}{2} I \right) \otimes \text{Tr}(e)^{\otimes 2n} \in C_{2n}(A).$$

The coefficient in front of this definition appears in the expression of the trace of an exponential matrix, the usual way to define the geometric Chern character with value in De Rham cohomology.

Lemma 3.5. *For any idempotent matrix $e \in M_r(A)$ and $n \geq 0$ the element $\text{Ch}_n(e)$ is closed, namely $(b + B)(\text{Ch}_n(e)) = 0$.*

This lemma ensures that the following definition is well-established.

Definition 3.6. *The **algebraic Chern character of a complex algebra** A is a group morphism defined as*

$$\begin{array}{ccc} \text{Ch}_{\text{alg}} : K_0(A) & \longrightarrow & HP_0(A) \\ [e] & \longrightarrow & \text{Ch}_{\text{alg}}(e) := (\text{Ch}_0(e), \text{Ch}_1(e), \dots) \end{array} .$$

This construction coincide with the geometric Chern character in the case of a commutative algebra [Lod92, p. 8.3.9], namely $\text{Ch} = \epsilon \circ \text{Ch}_{\text{alg}}$.

Although the Chern character is defined for a broad class of algebras, its existence does not guarantee that it provides useful information about their K-theory! The classes of algebras for which K-theory is well behaved and those for which periodic cyclic homology is informative do not necessarily coincide. For instance, consider a compact manifold M . On one hand, the continuous

inclusion of its Fréchet algebra of smooth functions $\mathcal{C}^\infty(M)$ into its C^* -algebra of continuous functions $\mathcal{C}^0(M)$ realizes an isomorphism in K-theory

$$K_0(\mathcal{C}^\infty(M)) \xrightarrow{\sim} K_0(\mathcal{C}^0(M)).$$

This is due to the De Rham theorem that relates the De Rham cohomology and the singular cohomology by integration [BT82]. On the other hand, whereas the periodic cyclic homology of $\mathcal{C}^\infty(M)$ is isomorphic to the De Rham cohomology of the manifold (see theorem 3.3), its C^* -algebra of continuous functions verifies

$$HP_0(\mathcal{C}^0(M)) \simeq \mathcal{B}(M) \quad \text{and} \quad HP_1(\mathcal{C}^0(M)) = 0$$

where $\mathcal{B}(M)$ is its space of Borel measures over M [Haa83]. While K-theory remains stable under these different completions, periodic cyclic theory drastically differentiate them; the situation can be summarized in the commutativity of the diagram

$$\begin{array}{ccc} K_0(\mathcal{C}^\infty(M)) & \xrightarrow{\sim} & K_0(\mathcal{C}^0(M)) \\ \text{Ch} \downarrow & & \downarrow \text{Ch} \\ HP_0(\mathcal{C}^\infty(M)) & \xrightarrow{\sim} & HP_0(\mathcal{C}^0(M)) \end{array} .$$

This example shows how periodic cyclic homology and K-theory can behave quite differently in certain situations and that the Chern character from K-theory to periodic cyclic homology can be far from being an isomorphism. To go further, the introduction of the book [Pus03] cite various examples of the Chern character's behavior depending on the completion-type of the algebra.

However, in some cases the Chern character becomes a powerful tool but after replacing the underlying algebra A by another one $\mathcal{A}$ with same K-groups but better computable periodic cyclic homology

$$\text{Ch} : K_0(A) \dashrightarrow K_0(\mathcal{A}) \longrightarrow HP_0(\mathcal{A}).$$

Since K-theory is stable under passage to dense subalgebras which are closed under holomorphic functional calculus (it is not the case for periodic cyclic homology), the main type of algebra one wants to work with are intermediate algebras between smooth and C^* -algebras, *e.g.* of Schwartz or Hecke types. They simultaneously retain and reflect information in the K-theoretical side and possesses enough analytic structure to be computable in periodic cyclic homology.

We list some examples where such strategy provides some informations.

Theorem 3.7 (Bradd-Kasparov). [Bra23] [Kas88] *Let G be a real reductive group. Its Casselman algebra $\mathfrak{S}(G)$ and its reduced C^* -algebra $C_r^*(G)$ are isomorphic in K -theory*

$$K_\bullet(\mathfrak{S}(G)) \xrightarrow{\sim} K_\bullet(C_r^*(G)).$$

Theorem 3.8 (Baum-Higson-Lafforgue-Plymen-Solleveld). [Sol05] [Sol07] [BHP00] [Laf06] *Let G be a reductive p -adic group. The Baum-Connes assembly map*

$$\mu : K_G^0(\mathcal{G}) \longrightarrow K_0(C_r^*(G))$$

from the equivariant K -theory of its Bruhat-Tits building $\mathcal{G}$ to the K -theory of its reduced C^ -algebra is rationally an isomorphism.*

Proof. (Just the main strategy). Write $\mathcal{H}(G)$ and $\mathcal{S}(G)$ for the Hecke and Schwartz algebras of the reductive p -adic group G . The proof follows from the fact that the following diagram commutes

$$\begin{array}{ccc} K_0(\mathcal{H}(G)) \simeq K_G^0(\mathcal{G}) & \xrightarrow{\mu} & K_0(C_r^*(G)) \simeq K_0(\mathcal{S}(G)) \\ \text{Ch} \downarrow & & \downarrow \text{Ch} \\ HP_0(\mathcal{H}(G)) & \xrightarrow{\iota} & HP_0(\mathcal{S}(G)) \end{array}$$

where the vertical Chern characters are rational isomorphisms and the bottom arrow is the continuous inclusion of the Hecke algebra into the Schwartz algebra, which is an isomorphism because these finite-type algebras are stratified equivalent. $\square$

Theorem 3.9 (G.). *Given a real reductive Lie group G with maximal compact subgroup K , there exists a Chern character*

$$\text{Ch} : K_\bullet(\mathcal{C}_\tau(G/K)) \simeq K_\bullet(\mathcal{C}_\tau^\infty(G/K)) \longrightarrow HP_\bullet(\mathcal{C}_\tau^\infty(G/K)) \simeq HP_\bullet(\mathcal{C}_c^\infty(G/K))$$

compatible with the left-translation action of the group G . Moreover, the product by the Chern character of the Dirac element $\not{D} \in K_{\dim(G/K)}^G(\mathcal{C}_\tau(G/K))$ realizes the degree-shift isomorphism

$$\text{Ch}(\not{D}) \times - : HP_\bullet(\mathcal{C}_c^\infty(K) \otimes_\pi \mathcal{D}) \xrightarrow{\sim} HP_{\bullet+\dim(G/K)}(\mathcal{C}_c^\infty(G) \otimes_\pi \mathcal{D})$$

after stabilization by a certain Schatten space $\mathcal{D}$.

The pairing of K -theory with periodic cyclic cohomology We define the periodic cyclic cohomology by dualizing (in the categorical meaning) the periodic cyclic complex and the differentials $b + B$. It leads to some cohomology groups $HP^0(A)$ and $HP^1(A)$ for any complete locally convex algebra A , the even part containing all the traces and higher traces over A . The periodic cyclic homology and cohomology groups can be paired via the natural evaluation morphism.

Proposition 3.10. *Let A be a complex algebra. There exists a complex valued pairing called **Chern-Connes pairing** between K -theory and periodic cyclic cohomology, given by*

$$\langle \cdot, \cdot \rangle : K_0(A) \times HP^0(A) \longrightarrow \mathbb{C}, \quad \langle [e], [\tau] \rangle := \tau(\text{Ch}(e)).$$

An important fact about this construction is that one can express Fredholm indexes using this Chern-Connes pairing [GE20].

Theorem 3.11. $\spadesuit$ (Connes-Moscovici) *Let A be a complex algebra and $(\mathcal{H}, F)$ a Fredholm module over it, with property $F^2 = 1$. There exists a periodic cyclic cocycle $[\tau_F] \in HP^0(A)$ such that for any idempotent matrix e over A the Fredholm index of the twisted Fredholm operator F_e can be expressed as*

$$\text{Index}(F_e) = \langle [e], [\tau_F] \rangle.$$

The computation of this pairing then provides expressions for topological invariants like Euler characteristic or higher signatures, but also some powerful statements in representation theory such as Selberg principle or Bass Trace conjecture. Some explicit computations in various cases have been performed by [HW18], [RWW20] and [ST19]. The question of invariance of this pairing while passing to a broader class of topological algebras is closely related to the injectivity of the Baum-Connes assembly map.

Is there a better homology than periodic cyclic homology? Although periodic cyclic homology provides a computable receptacle of the Chern character coming from K -theory, this construction might be disappointing for some classes of algebras. Other homologies such as asymptotic cyclic homology [Pus96] or local cyclic homology [Mey07] have been constructed to overcome this problem and propose alternative solutions in the C^* -algebraic case. As for periodic cyclic homology, they respect various stability and excision properties and are endowed with a Chern character. Moreover, they are preserved by restriction to adapted dense subalgebras and then faithfully reflect the properties of K -theory of C^* -algebras.

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